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ON THE NUMBER OF SOSOFs OF ORDER N

by

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ABSTRACT

Sum-of-squares orthogonal F-squares, SOSOFSs, of order n were introduced by Federer (2003a, 2003b). These SOSOFSs may be used to construct sum-of-squares orthogonal arrays for use in constructing codes and experiment designs. These arrays have varying numbers of symbols in the different rows of the arrays. Here we show how to compute the number of SOSOFSs in a complete set of sum-of-squares orthogonal F-squares.

Key words: Orthogonal array; sum-of-squares orthogonality; orthogonal Latin squares; geometrical interaction; degrees of freedom.

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INTRODUCTION

Sum-of-squares orthogonal F-squares, SOSOFSs, of order n were introduced by Federer (2003a, 2003b). These SOSOFSs may be used to construct sum-of-squares orthogonal arrays for use in constructing codes and experiment designs. These arrays have varying numbers of symbols in the different rows of the arrays. Here we show how to compute the number of SOSOFSs in a complete set of sum-of-squares orthogonal F-squares.

In the next section, it is shown how to enumerate the number of F-squares in a complete set of F-squares for n = p, p a prime number. It is shown how to construct Latin squares and F-squares using the geometrical interaction components. The construction of orthogonal arrays is also described. The number of F-squares for n = 2p is given in section three. In section four, the number of F-squares for $n = 2^k p$ is presented. The number of F-squares for $n = 2p^k$ is computed in section five. The previous results are generalized in section six for $n = 2^s p^k$. When n = pq, p and q different prime numbers, the number of F-squares with p symbols and the number of F-squares with q symbols in a complete set is given in section seven. This result is generalized in section eight for $n = p^s q^k$. Finally some comments are given in the last section.

$N = P^k$, P A PRIME NUMBER

Consider an $n \times n$ row by column arrangement where the rows are designated as factor A and the columns as factor B as for a two factor factorial. The row $\times$ column or $A \times B$ interaction has $(n - 1)^2$ degrees of freedom. When $n = p$, a prime number, the interaction may be partitioned into $p - 1$ geometrical interaction components AB, AB^2, ..., AB^{p-1}. Each of these components is associated with $p - 1$ degrees of freedom as there are p levels associated with each of them. Using the levels of a component as the treatments in a Latin square, this set may be used to construct a complete set of pairwise mutually orthogonal Latin squares, i. e., MOLS(p, p - 1). This type of orthogonality is combinatorial orthogonality, CO, in that every pair of symbols occurs with equal frequency.

A regular F-square is an $n \times n$ square with $p \leq n$ symbols where the p symbols occur n/p times in each row and in each column. It is denoted as F(n, n/p). A Latin square is the same as an F(n, 1). A semi-F-square, SF(n, n/p), is an $n \times n$ square where the p symbols occur n/p times in each row (column) but not all symbols occur in each column (row). When the sum of the sums of squares for the F-squares formed from an interaction adds to that for the interaction of factors, this is said to be sum-of-squares orthogonal, SOSO. For prime numbers, both CO and SOSO are both present.

For $p = 3$, the $p = 3$ levels of the Latin square constructed from the geometrical interaction component AB are obtained from 0 + 0, 0 + 1, 0 + 2, 1 + 0, 1 + 1, 1 + 2, 2 + 0, 2 + 1, and 2 + 2, modulus(3). The levels of AB^2 are obtained from 0 + 2(0), 0 + 2(1), 0 + 2(2), 1 + 2(0), 1 + 2 (1), 1 + 2(2), 2 + 2(0), 2 + 2(1), and 2 + 2(2), modulus(3). The rows are numbered 0, 1, ..., p - 1, and likewise for the columns. The MOLS(3, 2) set constructed from AB and AB^2 is:

0	1	2		0	2	1
1	2	0		1	0	2
2	0	1		2	1	0

When $n = p^2$, F-squares, $F(p^2, p)$, rather than Latin squares may be formed. The main effects and interactions are for a four-factor factorial. The row by column interaction has $(n - 1)^2 = (p^2 - 1)^2 = (p - 1)(p + 1)(p^2 - 1)$ degrees of freedom. There are $(p + 1)(p^2 - 1)$ geometrical interaction components each with $(p - 1)$ degrees of freedom. Each of these may be used to construct an $F(p^2, p)$-square. The $(p + 1)(p^2 - 1)$ $F(p^2, p)$ squares form a complete set and are CO as well as SOSO.

When $n = p^3$, F-squares, $F(p^3, p^2)$, are formed. For this case, a six-factor factorial is considered. The row by column interaction has $(n - 1)^2 = (p^3 - 1)^2 = (p - 1)(p^2 + p + 1)(p^3 - 1)$ degrees of freedom. There are $(p^2 + p + 1)(p^3 - 1)$ geometrical interaction components in the row by column interaction each with p symbols. Each one is associated with $p - 1$ degrees of freedom. This forms a complete set of F-squares with p symbols, and this set is CO and SOSO. This process may be continued for higher powers of prime numbers, p.

The numbers of such complete sets of F-squares are given below for p = 2, 3, 5, 7, and 11:

n	F-square	p = 2	p = 3	p = 5	p = 7	n=11
n = p	F(p, 1)	1	2	4	6	10
$n = p^2$	F(n, p)	9	32	144	384	1,440
$n = p^3$	$F(n, p^2)$	49	338	3,844	19,494	176,890

In general for $n = p^k$, there are $(n - 1)^2/(p - 1) = (p^k - 1)(p^{k-1} + p^{k-2} + \ldots + p^2 + p + 1)$ $F(p^k, p^{k-1})$ squares each having p symbols to form a complete set of CO and SOSO F-squares.

N = 2P, P A PRIME NUMBER

Let the rows of an n × n square be numbered 00, 01. ..., 0p-1, 10, 11, ..., 1p-1 and likewise for the columns. The row numbers represent the levels of a two-factor factorial for factors A and B, say, where A is at two levels and B at p levels. The column numbers are likewise represented as the levels of a 2 × p factorial for factors C and D. The interaction of A and C forms a F(n, p) square with two symbols and all the other interactions are used to construct regular F-squares or semi-F-squares with p symbols (Ferderer, 2003a, 2003b).

The degrees of freedom associated with the row by column interaction are $(n - 1)^2 = (2p - 1)^2$. One of these degrees of freedom is associated with the F(n, p) square with two symbols. Then $(2p - 1)^2 - 1 = 4p^2 - 4p = 4p(p - 1)$. Thus there are 4p F(n, 2) squares each of which is associated with p - 1 degrees of freedom. The set contains both regular F-squares and semi-F-squares. The numbers of F(n, 2) squares with p symbols for p = 3, 5, 7, and 11 are 12, 20, 28, and 44, respectively. The corresponding sum-of-squares orthogonal arrays that may be formed from these complete sets of SOSO sets are where SOSOA(n^2, number of symbols, number of F-squares):

n = 6:	SOSOA(36, 6, 2) + SOSOA(36, 2, 1) + SOSOA(36, 3, 12)
n = 10:	SOSOA(100, 10, 2) + SOSOA(100, 2, 1) + SOSOA(100, 5, 20)
n = 14	SOSOA(196, 14, 2) + SOSOA(196, 2, 1) + SOSOA(196, 7, 28)
n = 22	SOSOA(484, 22, 2) + SOSOA(484, 2, 1) + SOSOA(484, 11, 44)

Note that the rows and column symbols (numbers) form the first two rows of the array; the F-square with two symbols forms the third row of the array. The remaining 4p rows are formed from the 4p F(n, 2) squares. The array has width 2 + 1 + 4p = 4p + 3 rows.

N = 2^kP, P A PRIME NUMBER

Let the rows of an n × n square be numbered as for a $2^k \times p$ factorial. Number the columns in a similar fashion. This then forms a $2^{2k}p^2$ factorial arrangement. In the row by column interaction, there will be $2^{2k} - 1 - 2(2^k - 1) = (2^k - 1)^2$ interactions of the two level factors from rows and columns to form $(2^k - 1)^2$ $F(n, 2^{k-1}p)$ squares with two symbols. Then $(n - 1)^2 = (2^kp - 1)^2 = 2^{2k}p^2 - 2(2^kp) + 1 = (p - 1)(2^{2k}p + 2^{2k} - 2^{k+1}) + (2^k -$

$1)^2 = (p - 1)(2^k)(2^k p + 2^k - 2) + (2^k - 1)^2$. There are $2^k(2^k p + 2^k - 2)$ $F(n, 2^k)$ squares with p symbols. The numbers of F-squares for various values of n are:

n	k	p	$F(n, 2^{k-1}p)$	$F(n, 2^k)$
12	2	3	9	56
24	3	3	49	240
48	4	3	225	992
20	2	5	9	88
40	3	5	49	368
80	4	5	225	1,504
28	2	7	9	120
56	3	7	49	496
112	4	7	225	2,016

N= $2P^k$, P A PRIME NUMBER

The row by column interaction is associated with $(2p^k - 1)^2$ degrees of freedom. There is one $F(n, p^k)$ square with two symbols and the remaining are $F(n, 2p^{k-1})$ squares with p symbols. Now, $(2p^k - 1)^2 - 1 = 4p^{2k} - 4p^k = 4p^k(p^k - 1) = 4p^k(p - 1)(p^{k-1} + p^{k-2} + \dots + p + 1)$. Hence there are $4p^k(p^{k-1} + p^{k-2} + \dots + p + 1)$ $F(n, 2p^{k-1})$ squares with $p - 1$ degrees of freedom and p symbols each. When $p = 3$ and $k = 2$, $n = 18$. There is one F(18, 9) square and 36(4) = 144 F(18, 6) squares. The sum-of-squares orthogonal array formed from this set is SOSOA(324, 18, 2) + SOSOA(324, 2, 1) + SOSOA(324, 3, 144) with two rows of 18 symbols, one row of two symbols, and 144 rows of three symbols.

N = $2^s P^k$, P A PRIME NUMBER

The $(n - 1)^2 = (2^s p^k - 1)(2^s p^k - 1)$ degrees of freedom for the row by column interaction may be partitioned as follows for the various F-squares. $(2^s p^k - 1)(2^s p^k - 1) = (p - 1)[2^{2s}(p^{2k-1} + p^{2k-2} + \dots + p + 1) - 2^{s+1}(p^{k-1} + p^{k-2} + \dots + p + 1)] + (2^s - 1)^2$. There are $2^{2s}(p^{2k-1} + p^{2k-2} + \dots + p + 1) - 2^{s+1}(p^{k-1} + p^{k-2} + \dots + p + 1)]$ $F(n, 2^s p^{k-1})$ squares each with $p - 1$ degrees of freedom and p symbols. There are $(2^s - 1)^2$ $F(n, 2^{s-1}p^k)$ squares with two symbols.

N = PQ, P AND Q PRIME NUMBERS

The row by column interaction degrees of freedom may be partitioned as $(n - 1)^2 = (pq - 1)^2 = p^2q^2 - 2pq + 1 = (q - 1)(p^2q + p^2 - 2p) + (p - 1)^2$. Let $p = 3$ and $q = 5$. Then $196 = (5 - 1)[(9)(5) + 9 - 2(3)] + 2(2) = 4(48) + 2(2)$. There are 48 F(15, 3) squares with 5 symbols and two F(15, 5) squares with 3 symbols. In general, there are $p^2q + p^2 - 2p$ F(n, p) squares and $p - 1$ F(n, q) squares for $p < q$.

$N = P^sQ^k$, P AND Q PRIME NUMBERS

The row by column interaction degrees of freedom may be partitioned as follows $(n - 1)^2 = (p^sq^k - 1)^2 = (q - 1)[p^{2s}(q^{2k-1} + q^{2k-2} + \dots + q + 1) - 2p^s(q^{k-1} + q^{k-2} + \dots + q + 1)] + (p^s - 1)^2$. $(p^s - 1)^2 = (p - 1)[p^{2s-1} + p^{2s-2} + \dots + p + 1) - 2(p^{s-1} + p^{s-2} + \dots + p + 1)] = (p - 1)(p^s - 1)(p^{k-1} + p^{k-2} + \dots + p + 1)$. For $p < q$, there are $(p^s - 1)(p^{s-1} + p^{s-2} + \dots + p + 1)$ $F(n, p^{s-1}q^k)$ squares with p symbols. There are $p^{2s}(q^{2k-1} + q^{2k-2} + \dots + q + 1) - 2p^s(q^{k-1} + q^{k-2} + \dots + q + 1)$ $F(n, p^sq^{k-1})$ squares with q symbols.

COMMENTS

The number of SOSOFSs in a complete set has been determined for a number of situations. Using these methods, the number of SOSOFSs may be obtained for any number of products of prime numbers. As stated in Federer (2003a, 2003b), for p equal to a power of a prime number, complete sets of SOSOFSs have not been obtained for a power of p symbols, i. e., p^k symbols. Thus the projective geometry for prime powers is not included in the present geometry but the projective geometry for prime numbers is.

LITERATURE CITED

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